Instructions: You are not permitted to use a calculator of any kind on any portion of this exam. You are not allowed to use a textbook, notes, cell phone, computer, or any other technology on any portion of this exam. All devices must be turned off and stored away while you are in the examination room.

During this exam, any form of communication with any person other than the instructor or designated proctor is understood to be a violation of academic integrity.

No part of this exam may be removed from the examination room.

Read each question carefully. To receive full credit in the free response, you must:

- show legible, logical, and relevant justification which supports your final answer,
- use complete and correct mathematical notation,
- include proper units, where necessary, and
- give answers as exact values whenever possible.

You have ninety (90) minutes to complete this entire examination. Good luck!
Multiple Choice. On a piece of paper, write your answer selection to each of the following questions. You must clearly enumerate your responses. If no enumeration is provided, the grader will assume the ordering of your responses. Each question has only one correct answer, so if you indicate more than one answer, or leave a blank, the question is marked as incorrect. For this exam, there are fifteen (15) multiple choice questions worth three (3) points each for a total of forty-five (45) points.

1. Respond to this question with your exam version as indicated in the upper right-hand corner. 
   *This question has no point value.*
   
a. A  
b. B

2. The piecewise function given below is continuous at which of the following $x$-values?

   \[
   f(x) = \begin{cases} 
   2x - 5 & 0 \leq x \leq 2 \\
   -5x + 9 & 2 < x < 3 \\
   2x - 8 & 3 \leq x \leq 4 
   \end{cases}
   \]

   a. $x = 3$  
b. $x = 2, x = 3$  
c. $x = 2$

   \[
f(2) \in \mathbb{R} \text{ and } \lim_{x \to 2} f(x) \text{ exists with } \lim_{x \to 2} f(x) = f(2)
   \]

3. In order to obtain an inverse trigonometric function, a domain restriction was imposed. The purpose of the domain restriction was to enforce which of the following properties of functions?

   a. continuity  
b. one-to-one  
c. onto

4. Evaluate the limit: \( \lim_{x \to 0^+} \frac{2x - 1}{x^2} \).

   a. $-\infty$  
b. $+\infty$

   \[
   \lim_{x \to 0^+} \frac{2x - 1}{x^2} = \frac{-1}{\text{small}^+} = -\infty
   \]
5. Consider the following numerical data.

<table>
<thead>
<tr>
<th>x</th>
<th>0.9000</th>
<th>0.9900</th>
<th>0.9990</th>
<th>0.9999</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>0.5132</td>
<td>0.5013</td>
<td>0.5001</td>
<td>0.5000</td>
</tr>
</tbody>
</table>

Which of the following statements does the given numerical data support?

a. \( \lim_{x \to 1^+} f(x) = \frac{1}{2} \)

b. \( \lim_{x \to 1^-} f(x) = \frac{1}{2} \)

c. \( \lim_{x \to 1} f(x) = \frac{1}{2} \)

6. Find the value of constant \( k \) so that \( \lim_{x \to \infty} k \arctan x = 1 \).

a. \( k = \frac{\pi}{2} \)

b. \( k = -\frac{2}{\pi} \)

c. \( k = \frac{2}{\pi} \)

\[
\lim_{x \to \infty} k \arctan x = 1 \iff k \lim_{x \to \infty} \arctan x \frac{\pi}{2} \implies k \cdot \frac{\pi}{2} = 1 \implies k = \frac{2}{\pi}
\]

7. Which of the following expressions equals \( \frac{3\pi}{4} \)?

a. \( \arcsin \left( \sin \left( \frac{3\pi}{4} \right) \right) \)

b. \( \arccos \left( \cos \left( \frac{5\pi}{4} \right) \right) \)

c. \( \arccos \left( \sin \left( \frac{3\pi}{4} \right) \right) \)
8. Suppose \( f(x) = x^2 - 3x + 2 \). The slope of the tangent line at \( x = \frac{3}{2} \) is ... 

a. 0  
b. \(-\frac{1}{4}\)  
c. 3 

\[ f'(x) = 2x - 3 \implies f'(\frac{3}{2}) = 2 \cdot \frac{3}{2} - 3 = 0 \]

9. Find the third derivative of \( f(x) = x^3 - 2e^x \).

a. \( f^{(3)}(x) = 6 - 2e^x \)  
b. \( f^{(3)}(x) = 6x - 2e^x \)  
c. \( f^{(3)}(x) = 6 - e^x \)

\[ f'(x) = 3x^2 - 2e^x \implies f''(x) = 6x - 2e^x \implies f^{(3)}(x) = 6 - 2e^x \]

10. Which of the following three numbers is the largest?

a. \( \log_5 \sqrt[3]{5} \)  
b. \( \log_5 1 \)  
c. \( \log_5 \sqrt[5]{5} \)

\[ \log_5 \sqrt[3]{5} = \log_5 5^{\frac{1}{3}} = \frac{1}{3} \quad \log_5 1 = 0 \quad \log_5 \sqrt[5]{5} = \log_5 5^{\frac{1}{5}} = \frac{1}{2} \]
11. Given that \( \tan \theta = \frac{1}{3} \), find \( \cos \theta \) if the angle \( \theta \) lies in the third quadrant.

   a. \( \cos \theta = -\frac{1}{\sqrt{10}} \)  
   b. \( \cos \theta = -\frac{3}{\sqrt{10}} \)  
   c. \( \cos \theta = \frac{3}{\sqrt{10}} \)

12. Use the graph of the function \( f \) below to calculate the average rate of change on the interval \([0, 3]\).

   ![Graph of the function f](image)

   a. \(-1\)  
   b. \(1\)  
   c. \(-3\)

   \[
   \frac{f(3) - f(0)}{3 - 0} = \frac{0 - 3}{3 - 0} = -1
   \]

13. Evaluate the limit: \( \lim_{x \to -\infty} \frac{2x - 3}{\sqrt{4x^2 + 1}} \).

   a. \(1\)  
   b. \(-1\)  
   c. \(0\)

   \[
   \lim_{x \to -\infty} \frac{2x - 3}{\sqrt{4x^2 + 1}} = \lim_{x \to -\infty} \frac{2x}{\sqrt{4x^2 + 1}} - \lim_{x \to -\infty} \frac{3}{\sqrt{4x^2 + 1}} = \lim_{x \to -\infty} \frac{2x}{\sqrt{4 + \frac{1}{x^2}}} - \lim_{x \to -\infty} \frac{3}{\sqrt{4 + \frac{1}{x^2}}} = \lim_{x \to -\infty} \frac{-2 + \frac{0}{x^2}}{\sqrt{4 + \frac{0}{x^2}}} = \frac{-2}{\sqrt{4}} = -1
   \]
14. Find the derivative of \( y = e^x - \frac{1}{2}x^2 + 2\sqrt{x^3} \).

a. \( y' = e^x - x + 3\sqrt{x} \)

b. \( y' = xe^{x-1} - x + 3\sqrt{x} \)

c. \( y' = e^x - x + \frac{4}{3\sqrt{x}} \)

\[
y = e^x - \frac{1}{2}x^2 + 2x^{\frac{3}{2}} \implies y' = e^x - x + 3x^{\frac{3}{2}} = e^x - x + 3\sqrt{x}
\]

15. Since \( f(x) = 2x^2 - x - 3 \) is continuous for all real numbers, the Intermediate Value Theorem guarantees that \( f(x) = 0 \) for some \( x \)-value contained within which of the following intervals?

a. \( (0, 1) \)

b. \( (0, 2) \)

c. \( (0, \frac{1}{2}) \)

\[
f(0) = -3, \ f(2) = 3 \implies f(x) = 0, \ x \in (0, 2) \text{ since } f \text{ is cts on } [0, 2]
\]

16. Find the \( (x, y) \) pair on the graph of \( f(x) = e^x + 2x \) where the slope of the tangent line is three (3).

a. \( (1, e + 2) \)

b. \( (0, 3) \)

c. \( (0, 1) \)

\[
f'(x) = e^x + 2 \implies e^x + 2 = 3 \iff e^x = 1 \implies x = 0
\]

\[
x = 0 \implies y = f(0) = e^0 + 0 = 1 \implies (0, 1)
\]

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This concludes the multiple choice portion of this exam.

Please take a moment to verify that you have sixteen (16) total responses, with the first response holding your exam version, Version A.
Free Response. Answer all questions clearly and legibly on blank paper. If your work is illegible, it may be marked incorrect. Additionally, correct answers without appropriate work shown will receive little or no credit. For each question, show specific mathematical notation, work, and any appropriate units. In addition, you may not use L'Hôpital’s Rule on this portion of the exam. Before you begin, read the directions at the beginning of the exam. This portion of the exam is worth fifty-five (55) points.

1. (16 pts) Evaluate the following limits, showing all work. Remember that your notation is also graded. Absolutely no credit will be awarded for using L'Hôpital’s Rule.

   a. (4 pts) \( \lim_{x \to 0} \frac{\sqrt{3x + 9} - 3}{x} \)

      \[
      \lim_{x \to 0} \frac{\sqrt{3x + 9} - 3}{x} = \lim_{x \to 0} \frac{\sqrt{3x + 9} - 3}{x} \cdot \frac{\sqrt{3x + 9} + 3}{\sqrt{3x + 9} + 3}
      \]
      multiply by the conjugate

      \[
      = \lim_{x \to 0} \frac{\sqrt{3x + 9} \sqrt{3x + 9} - 3 \cdot 3}{x(\sqrt{3x + 9} + 3)}
      \]

      \[
      = \lim_{x \to 0} \frac{3x + 9 - 9}{x(\sqrt{3x + 9} + 3)}
      \]

      \[
      = \lim_{x \to 0} \frac{3x}{x(\sqrt{3x + 9} + 3)}
      \]

      \[
      = \lim_{x \to 0} \frac{3}{\sqrt{3x + 9} + 3}
      \]

      \[
      = \frac{1}{2}
      \]

   b. (4 pts) \( \lim_{x \to 3} \frac{x^2 - 4x + 3}{x^2 - 9} \)

      \[
      \lim_{x \to 3} \frac{x^2 - 4x + 3}{x^2 - 9} = \lim_{x \to 3} \frac{(x - 3)(x - 1)}{(x - 3)(x + 3)} = \lim_{x \to 3} \frac{x - 1}{x + 3} = \frac{1}{3}
      \]
      factor and cancel
c. (4 pts) \(\lim_{{x \to \infty}} \frac{3x^3 + 2x^2 + x}{2x^3 + 9x - 1}\)

\[
\lim_{{x \to \infty}} \frac{3x^3 + 2x^2 + x}{2x^3 + 9x - 1} = \lim_{{x \to \infty}} \frac{\frac{3x^3}{x^3} + \frac{2x^2}{x^3} + \frac{x}{x^3}}{\frac{2x^3}{x^3} + \frac{9x}{x^3} - \frac{1}{x^3}} = \lim_{{x \to \infty}} \frac{3 + \frac{2}{x^2} + \frac{1}{x^3}}{2 + \frac{9}{x^2} - \frac{1}{x^3}} = \frac{3}{2}
\]

*divide by the highest power in the denominator*

---

d. (4 pts) \(\lim_{{x \to \pi}} \frac{1}{\pi} - \frac{1}{x}\)

\[
\lim_{{x \to \pi}} \frac{1}{\pi} - \frac{1}{x} = \lim_{{x \to \pi}} \frac{x - \pi}{\pi(x - \pi)} = \lim_{{x \to \pi}} \frac{\pi}{\pi x} \cdot \frac{1}{\pi} = \lim_{{x \to \pi}} \frac{1}{\pi x} = \frac{1}{\pi^2}
\]

*find a common denominator*
2. (11 pts) Apply the limit definition of the derivative to find the derivative of \( f(x) = \sqrt{4x - 3} \).

You must show all work. The use of derivative rules will result in no credit.

\[
\begin{align*}
  f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
  &= \lim_{h \to 0} \frac{\sqrt{4(x+h) - 3} - \sqrt{4x - 3}}{h} \\
  &= \lim_{h \to 0} \frac{\sqrt{4x + 4h - 3} - \sqrt{4x - 3}}{h} \cdot \frac{\sqrt{4x + 4h - 3} + \sqrt{4x - 3}}{\sqrt{4x + 4h - 3} + \sqrt{4x - 3}} \\
  &= \lim_{h \to 0} \frac{4x + 4h - 3 - (4x - 3)}{h(\sqrt{4x + 4h - 3} + \sqrt{4x - 3})} \\
  &= \lim_{h \to 0} \frac{4h}{h(\sqrt{4x + 4h - 3} + \sqrt{4x - 3})} \\
  &= \lim_{h \to 0} \frac{4}{\sqrt{4x + 4h} - 3 + \sqrt{4x - 3}} \\
  &= \frac{4}{\sqrt{4x - 3} + \sqrt{4x - 3}} \\
  &= \frac{2}{2\sqrt{4x - 3}} \\
  &= \frac{1}{\sqrt{4x - 3}}
\end{align*}
\]
3. (10 pts) In this question, consider the given rational function.

\[ f(x) = \frac{3x^2 - 3x - 1}{x - 2} \]

a. (6 pts) Find the equation of the horizontal and/or oblique (slant) asymptote of \( f \).

Since this is a “top-heavy” rational function, there is an oblique asymptote. By polynomial long division, we find its equation.

\[
\begin{array}{c|cccc}
 & 3x & + 3 \\
\hline 
3x - 2 & 3x^2 & -3x & -1 \\
 & 3x^2 & -6x \\
\hline 
& 3x & -1 \\
 & 3x & +6 \\
\hline 
& 5 
\end{array}
\]

Hence, the equation of the oblique (slant) asymptote is \( y = 3x + 3 \). Recall that horizontal and oblique asymptotes may not coexist.

b. (4 pts) Find and verify (using limits) the equation of the vertical asymptote of \( f \).

We claim that \( x = 2 \) is a vertical asymptote of \( f \). Next, we verify with a limit.

\[
\lim_{x \to 2^+} \frac{3x^2 - 3x - 1}{x - 2} = \frac{5}{\text{small}^+} = +\infty
\]

\[
\lim_{x \to 2^-} \frac{3x^2 - 3x - 1}{x - 2} = \frac{5}{\text{small}^-} = -\infty
\]

Either of these limits is sufficient to demonstrate that \( x = 2 \) is a vertical asymptote of \( f \).
4. **(8 pts)** Refer to the graph of the function $f$ below for each of the following questions. Assume that the graph continues outside of the region shown.

![Graph of the function f](image)

a. **(1 pts)** Calculate the value of $f(0) - f(-2)$.

$$f(0) - f(-2) = 0 - 2 = -2$$

b. **(1 pts)** Evaluate $\lim_{x \to -2} f(x)$.

$$\lim_{x \to -2} f(x) = 1$$

c. **(1 pts)** What type of discontinuity is demonstrated at $x = -2$?

removable discontinuity

d. **(1 pts)** Evaluate $\lim_{x \to 0} f(x)$.

$$\lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x) \implies \lim_{x \to 0} f(x) \text{ does not exist (DNE)}$$

e. **(1 pts)** Evaluate $\lim_{x \to -3^+} f(x)$.

$$\lim_{x \to -3^+} f(x) = -\infty$$
f. **(1 pts)** Evaluate \( \lim_{x \to \infty} f(x) \).

\[
\lim_{x \to \infty} f(x) = -2
\]

g. **(2 pts)** List all \( x \)-values where \( f \) is not differentiable.

\[ x = -3, -2, -1, 0 \]

5. **(10 pts)** Consider the function \( f(x) = \frac{1}{2}x^2 - 2x \).

a. **(7 pts)** Find \( f'(1) \) using the form \( f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \).

\[
f'(1) = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h}
= \lim_{h \to 0} \frac{\frac{1}{2}(1 + h)^2 - 2(1 + h) - \left(-\frac{3}{2}\right)}{h}
= \lim_{h \to 0} \frac{\frac{1}{2}(1 + h^2 + 2h) - 2 - 2h + \frac{3}{2}}{h}
= \lim_{h \to 0} \frac{\frac{1}{2}h^2 + \frac{1}{2}h + 2 - 2h + \frac{3}{2}}{h}
= \lim_{h \to 0} \frac{\frac{1}{2}h^2 - h}{h}
= \lim_{h \to 0} \frac{h(\frac{1}{2}h - 1)}{h}
= \lim_{h \to 0} \left( \frac{1}{2}h - 1 \right)
= -1
\]

b. **(3 pts)** Write the equation of the tangent line at \( x = 1 \) in point-slope form.

From (a), we have that \( m_{\text{tan}} = f'(1) = -1 \). When \( x = 1 \), \( y = f(1) = -\frac{3}{2} \), so we are now able to assemble the equation of the tangent line at \( x = 1 \).

\[
y - \left( -\frac{3}{2} \right) = -1(x - 1)
\]
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